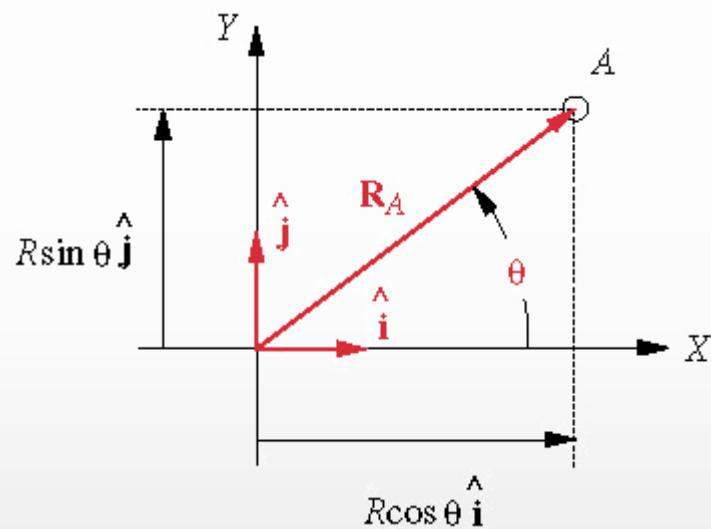


# **Kinematics & Dynamics of Linkages**

Lecture II – Analytical Linkage Synthesis

# Algebraic analysis

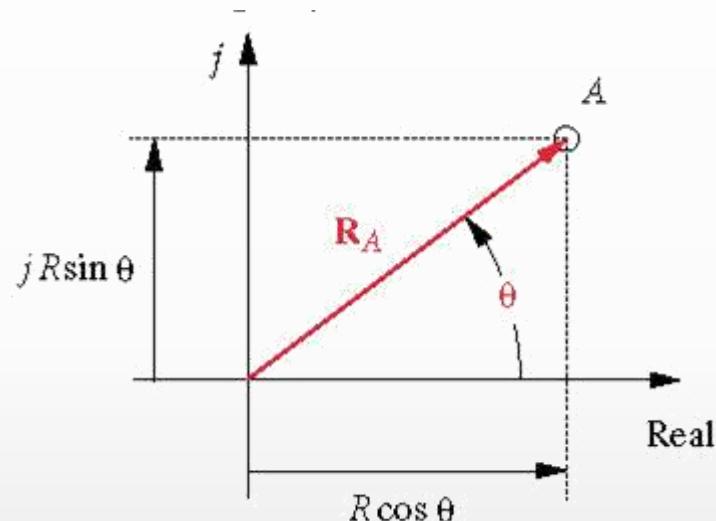


Polar form:

$$| \mathbf{R}_A | @ \angle \theta$$

Cartesian form:

$$R\cos \theta \hat{\mathbf{i}}, R\sin \theta \hat{\mathbf{j}}$$



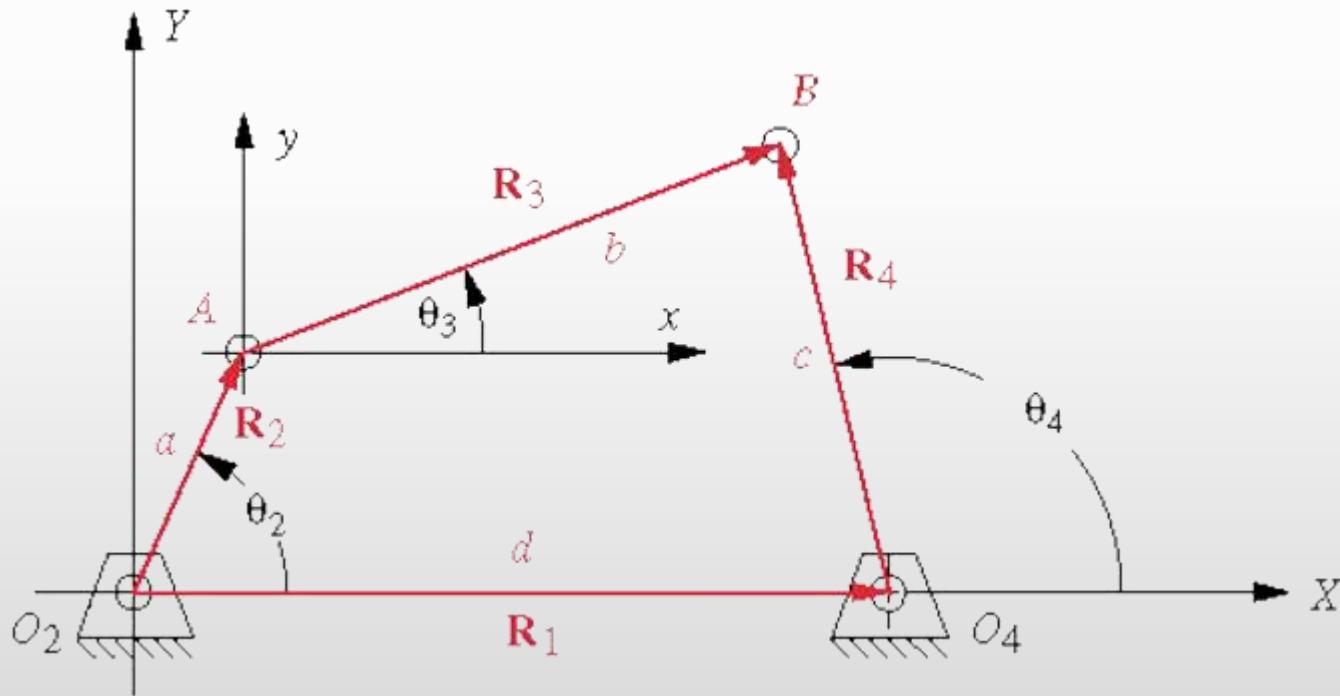
Polar form:  $R e^{j\theta}$

Cartesian form:  $R \cos \theta + j R \sin \theta$

$$R = | \mathbf{R}_A |$$

# Vector Loop Approach

$$R_2 + R_3 - R_4 - R_1 = 0$$



# Equations

$$R_2 + R_3 - R_4 - R_1 = 0$$

**Real part**

$$a \cos\theta_2 + b \cos\theta_3 - c \cos\theta_4 - d = 0$$

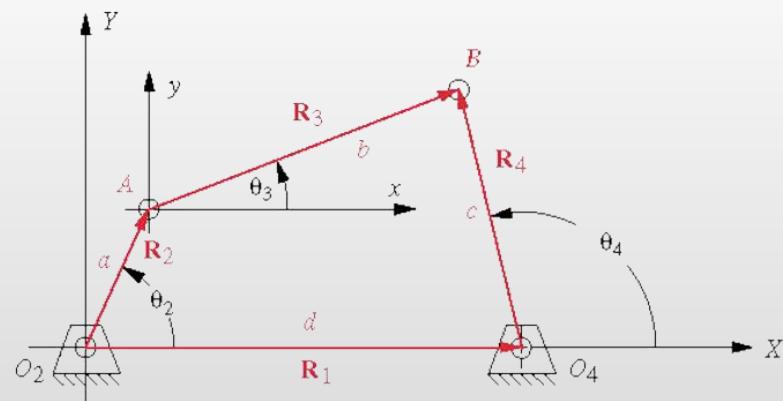
**Imaginary part**

$$a \sin\theta_2 + b \sin\theta_3 - c \sin\theta_4 = 0$$

**Goal is to find**

$$\theta_3 = f(a, b, c, d, \theta_2)$$

$$\theta_4 = g(a, b, c, d, \theta_2)$$



# 4Bar Solution Derivation

**Isolate  $\Theta_3$**

$$b \cos\Theta_3 = -a \cos\Theta_2 + c \cos\Theta_4 + d \quad b \sin\Theta_3 = -a \sin\Theta_2 + c \sin\Theta_4$$

**Square both sides**

$$(b \cos\Theta_3)^2 = (-a \cos\Theta_2 + c \cos\Theta_4 + d)^2 \quad (b \sin\Theta_3)^2 = (-a \sin\Theta_2 + c \sin\Theta_4)^2$$

**Add the 2 expressions**

$$b^2 (\cos^2\Theta_3 + \sin^2\Theta_3) = (-a \cos\Theta_2 + c \cos\Theta_4 + d)^2 + (-a \sin\Theta_2 + c \sin\Theta_4)^2$$

**So this yields** (using  $\cos^2\Theta_3 + \sin^2\Theta_3 = 1$ )

$$b^2 = (-a \cos\Theta_2 + c \cos\Theta_4 + d)^2 + (-a \sin\Theta_2 + c \sin\Theta_4)^2$$

# 4Bar Solution Derivation

Multiply & combine like terms

$$b^2 = a^2 + c^2 + d^2 - 2ad \cos\theta_2 + cd \cos\theta_4 - 2ac \cos(\theta_2 - \theta_4)$$

Divide both sides by 2ac

$$-\frac{(b^2+a^2+c^2+d^2)}{2ac} + \left(\frac{d}{c}\right) \cos\theta_2 - \left(\frac{d}{a}\right) \cos\theta_4 = -\cos(\theta_2 - \theta_4)$$

To simplify, define

$$k_1 = \left(\frac{d}{a}\right)$$

$$k_2 = \left(\frac{d}{c}\right)$$

$$k_3 = \frac{(a^2 - b^2 + c^2 + d^2)}{2ac}$$

Yields Freudenstein's equation

$$k_1 \cos\theta_4 - k_2 \cos\theta_2 + k_3 = \cos(\theta_2 - \theta_4) = \cos\theta_2 \cos\theta_4 + \sin\theta_2 \sin\theta_4$$

# 4Bar Solution Derivation

Yields Freudenstein's equation

$$k_1 \cos \theta_4 - k_2 \cos \theta_2 + k_3 = \cos \theta_2 \cos \theta_4 + \sin \theta_2 \sin \theta_4$$

From Trigonometric relations:

$$\sin \theta_4 = (2 \tan(\theta_4/2)) / (1 + \tan^2(\theta_4/2))$$

$$\cos \theta_4 = (1 - \tan^2(\theta_4/2)) / (1 + \tan^2(\theta_4/2))$$

Substituting      A  $\tan^2(\theta_4/2)$  + B  $\tan(\theta_4/2)$  + C = 0

where

$$A = \cos \theta_2 - k_1 - k_2 \cos \theta_2 + k_3$$

$$B = -2 \sin \theta_2$$

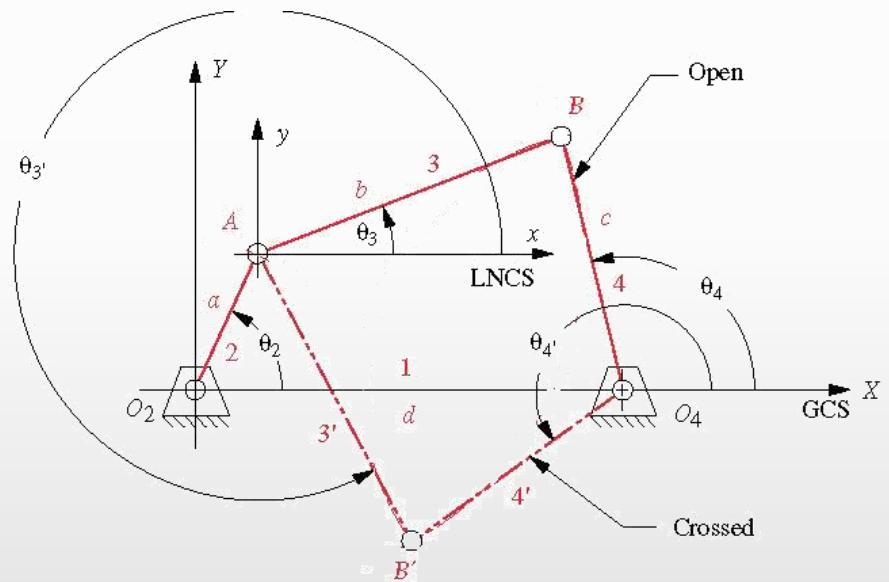
$$C = k_1 - (k_2 + 1) \cos \theta_2 + k_3$$

# 4Bar Solution Derivation

Solution of  $A \tan^2(\theta_4/2) + B \tan(\theta_4/2) + C = 0$

$$\tan\left(\frac{\theta_4}{2}\right) = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}$$

$$\theta_4 = 2 \tan^{-1}\left(\frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}\right)$$



We have **2 solutions** because we have 2 possible linkage configurations (**Open** and **Crossed**)

# 4Bar Solution Derivation

Solving for  $\theta_3$ :

Original loop equations

$$a \cos\theta_2 + b \cos\theta_3 - c \cos\theta_4 - d = 0 \quad a \sin\theta_2 + b \sin\theta_3 - c \sin\theta_4 = 0$$

Isolate  $\theta_4$

$$c \cos\theta_4 = a \cos\theta_2 + b \cos\theta_3 - d \quad c \sin\theta_4 = a \sin\theta_2 + b \sin\theta_3$$

Square both sides and add the equations to eliminate  $\theta_4$

$$k_1 \cos\theta_3 - k_4 \cos\theta_2 + k_5 = \cos(\theta_2 - \theta_3) = \cos\theta_2 \cos\theta_3 + \sin\theta_2 \sin\theta_3$$

Where

$$k_1 = (d/a)$$

$$k_4 = (d/b)$$

$$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab$$

# 4Bar Solution Derivation

Similarly, we may derive a quadratic equation for  $\tan(\theta_3/2)$

$$D \tan^2(\theta_3/2) + E \tan(\theta_3/2) + F = 0$$

Where

$$D = \cos\theta_2 - k_1 + k_4 \cos\theta_2 + k_5$$

$$E = -2 \sin\theta_2$$

$$F = k_1 + (k_4 - 1) \cos\theta_2 + k_5$$

$$k_1 = (d/a)$$

$$k_4 = (d/b)$$

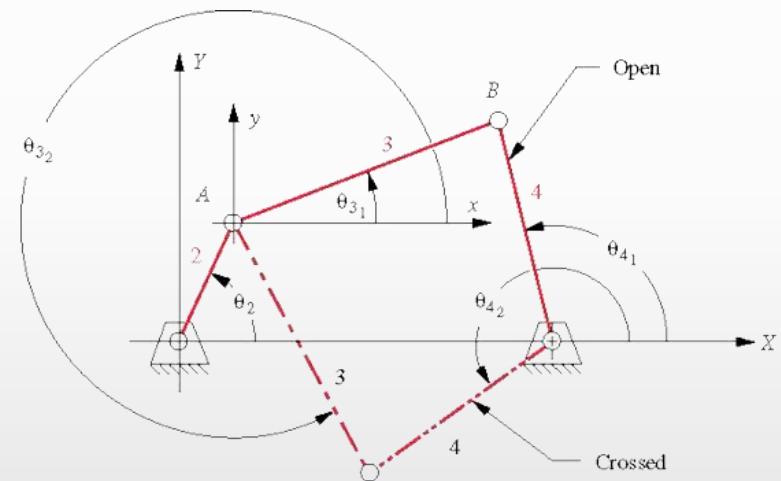
$$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab$$

# 4Bar Solution Derivation

The solution to the quadratic is

$$\tan\left(\frac{\theta_3}{2}\right) = \frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D}$$

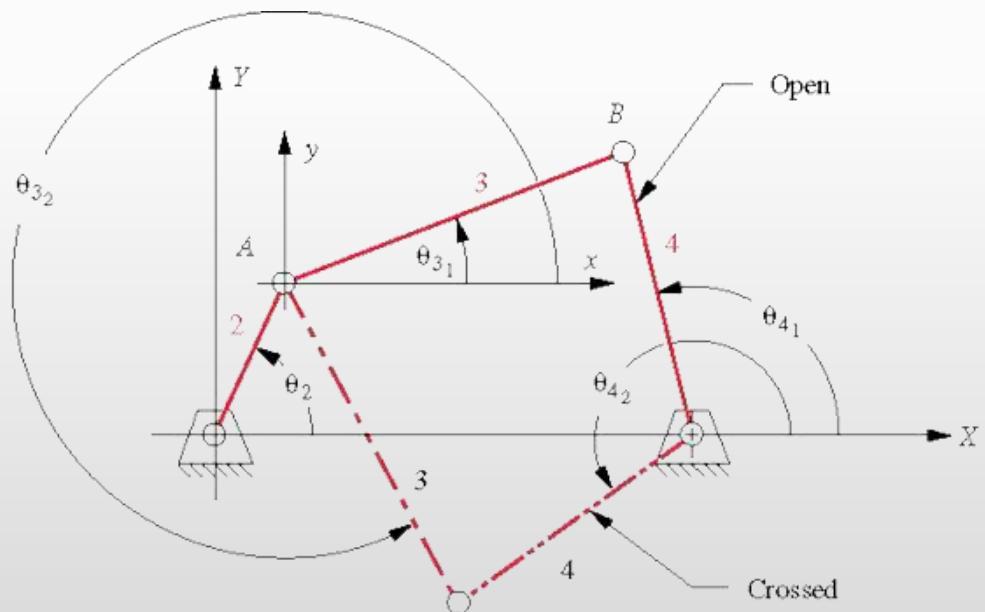
$$\theta_3 = 2 \tan^{-1} \left( \frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D} \right)$$



Again, 2 solutions for open & crossed linkages

# Example

- Link 1 = 8"
- Link 2 = 5"
- $\theta_2 = 75^\circ$
- Calculate  $\theta_3$  &  $\theta_4$  ??



# Example - Solution

To determine  $\theta_3$  &  $\theta_4$  first calculate the constants

$$k_1 = (d/a) = 1.6$$

$$k_2 = (d/c) = 1.333$$

$$k_3 = (a^2 - b^2 + c^2 + d^2)/2ac = 1.017$$

$$k_4 = (d/b) = 1.0$$

$$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab = -1.463$$

*a = length of link 2*

*b = length of link 3*

*c = length of link 4*

*d = length of link 1*

$$A = \cos\theta_2 - k_1 - k_2 \cos\theta_2 + k_3 = -0.669$$

$$B = -2\sin\theta_2 = -1.932$$

$$C = k_1 - (k_2 + 1)\cos\theta_2 + k_3 = 2.013$$

$$D = \cos\theta_2 - k_1 + k_4 \cos\theta_2 + k_5 = -2.545$$

$$E = -2\sin\theta_2 = -1.932$$

$$F = k_1 + (k_4 - 1)\cos\theta_2 + k_5 = 0.137$$

# Example - Solution

The solution of  $\theta_4$ :

$$\theta_4 = 2 \tan^{-1} \left( \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A} \right) = 78.2^\circ \text{ or } -149.7^\circ$$

The solution of  $\theta_4$ :

$$\theta_3 = 2 \tan^{-1} \left( \frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D} \right) = 7.5^\circ \text{ or } -79.0^\circ$$

# Slider-Crank

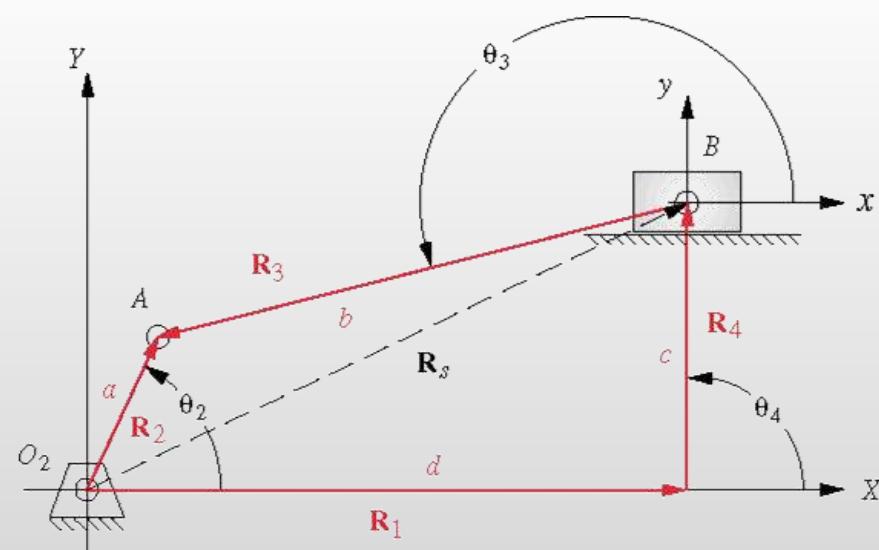
## Vector Loop Approach

Given:

- Lengths of links 2 & 3
- Offset height
- Input angle  $\theta_2$

Find:

- Length of link 1
- Angle  $\theta_3$



# Slider-Crank

Vector loop equation

$$R_2 - R_3 - R_4 - R_1 = 0$$

Projections: (x and y axis)

$$a \cos\theta_2 - b \cos\theta_3 - c \cos\theta_4 - d = 0$$

$$a \sin\theta_2 - b \sin\theta_3 - c \sin\theta_4 = 0$$

Where

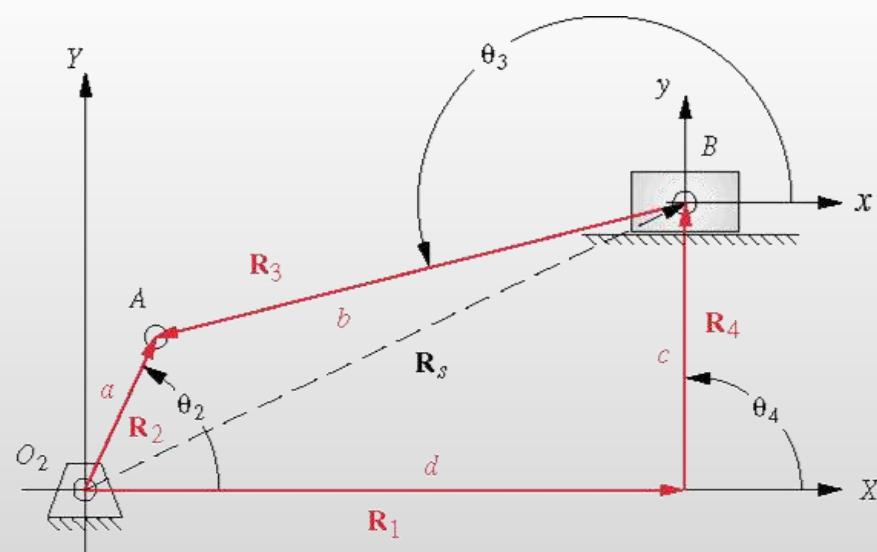
a, b and c are known

$\theta_2$  is given

$\theta_4 = 90^\circ$

Solution

$$d = a \cos\theta_2 - b \cos\theta_3$$



# Slider-Crank

Solve for  $\theta_3$  using the equation (projection on y axis)

$$a \sin \theta_2 - b \sin \theta_3 - c \sin \theta_4 = 0$$

Solution ( $\pm 90^\circ$  of  $\sin^{-1}$ )

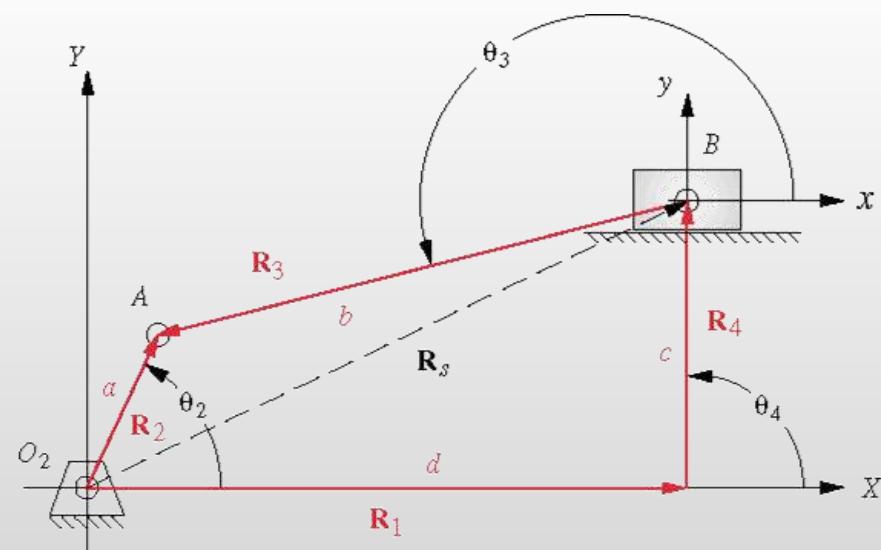
1<sup>st</sup> configuration

$$\theta_{3_1} = \sin^{-1} \left( \frac{a \sin \theta_2 - c}{b} \right)$$

2<sup>nd</sup> configuration

$$\theta_{3_2} = \sin^{-1} \left( -\frac{a \sin \theta_2 - c}{b} \right) + \pi$$

$$d = a \cos \theta_2 - b \cos \theta_3$$



# Inverted Slider-Crank - Sliding joint between links 3 & 4 at point B - $\gamma$ does not have to be $90^\circ$ - All slider-crank will have at least one link whose effective length between joints will vary as the linkage moves The diagram illustrates an inverted slider-crank mechanism in a 2D coordinate system with a horizontal X-axis and a vertical Y-axis. Link 1 is the fixed base, represented by a dashed line segment from the origin O<sub>1</sub> to point A. Link 2 is the coupler, a red line segment connecting A to joint 2, which is pinned to the base. Link 3 is the crank, a red line segment connecting joint 2 to joint B. Link 4 is the slider, a grey rectangular block pivoted at joint B. Link 4 is constrained to move linearly along the Y-axis. The angle between link 2 and the horizontal is $\theta_2$ . The angle between link 3 and the horizontal is $\theta_3$ . The angle between link 4 and the Y-axis is $\theta_4$ . The distance from A to joint 2 is labeled $R_B$ . The angle $\gamma$ is the angle between the Y-axis and the projection of link 3 onto the Y-axis. Dashed lines indicate the projections of joints 2 and B onto the X-axis. MEE341 - Lecture 11: Analytical Linkage Synthesis Slide 18 of 26 LAU

# Inverted Slider-Crank - Vector Loop Approach - Length of link 3 ( $b$ ) changes with time - $b$ is unknown - $\theta_4$ is unknown - $\theta_3$ is unknown - Relationship between $\theta_3$ and $\theta_4$ $$\theta_3 = \theta_4 + \gamma$$ - + for an open linkage - for a crossed linkage The diagram illustrates an inverted slider-crank mechanism in a 2D coordinate system with a horizontal X-axis and a vertical Y-axis. The mechanism consists of four links labeled 1 through 4. Link 1 is the fixed base. Link 2 is the coupler, connected to link 1 at joint $Q_2$ and to link 3 at joint $Q_3$ . Link 3 is the slider, connected to link 2 at joint $Q_3$ and to link 4 at joint $Q_4$ . Link 4 is the crank, connected to link 2 at joint $Q_2$ and to the slider at joint $Q_4$ . The slider 3 moves along the Y-axis. The crank 4 rotates about joint $Q_2$ . The angle between the coupler 2 and the slider 3 is $\theta_2$ . The angle between the slider 3 and the crank 4 is $\theta_3$ . The angle between the coupler 2 and the crank 4 is $\theta_4$ . The angle between the Y-axis and the slider 3 is $\gamma$ . A coordinate system is centered at the slider 3, with the Y-axis pointing upwards and the X-axis pointing to the right. The position of the slider 3 is defined by its coordinates $(x, y)$ . MEE341 - Lecture 11: Analytical Linkage Synthesis Slide 19 of 26 LAU

# Inverted Slider-Crank

**Vector loop equation:**  $R_2 - R_3 - R_4 - R_1 = 0$

**Original loop equations**

$$a \cos\theta_2 - b \cos\theta_3 - c \cos\theta_4 - d = 0$$

$$a \sin\theta_2 - b \sin\theta_3 - c \sin\theta_4 = 0$$

**Where** a, d and c are known,  $\theta_2$  is given,  $\theta_4$ ,  $\theta_3$  and b to be calculated

**Use the second equation to solve for b**

$$b = (a \sin\theta_2 - c \sin\theta_4) / \sin\theta_3$$

**Substituting into the first equation**

$$a \cos\theta_2 - ((a \sin\theta_2 - c \sin\theta_4) / \sin\theta_3) \cos\theta_3 - c \cos\theta_4 - d = 0$$

# Inverted Slider-Crank Rearrange equation $$a \cos\theta_2 - ((a \sin\theta_2 - c \sin\theta_4) / \sin\theta_3) \cos\theta_3 - c \cos\theta_4 - d = 0$$ Using: $\theta_3 = \theta_4 \pm \gamma$ Yields: $P \sin\theta_4 + Q \cos\theta_4 + R = 0$ Where $P = a \sin\theta_2 \sin\gamma + (a \cos\theta_2 - d) \cos\gamma$ $Q = -a \sin\theta_2 \cos\gamma + (a \cos\theta_2 - d) \sin\gamma$ $R = -c \sin\gamma$ The diagram illustrates the geometry of an inverted slider-crank mechanism. It features a fixed pivot at joint 1 (bottom left), a connecting rod 2 (link AB), a coupler link 3 (link BC), and a slider block 4 (link CD). Joint 2 is at a distance $R_B$ from the center of rotation at joint 1. Joint 3 is at a distance $R_C$ from joint 2. Joint 4 is a slider moving in a vertical slot. The angle $\theta_2$ is the angle between the horizontal projection of the connecting rod and the horizontal axis. The angle $\theta_3$ is the angle between the coupler link and the horizontal projection of the connecting rod. The angle $\theta_4$ is the angle between the coupler link and the horizontal projection of the slider block. The angle $\gamma$ is the angle between the coupler link 3 and the slider block 4. A coordinate system (x, y) is centered at joint 1. MEE341 – Lecture 11: Analytical Linkage Synthesis Slide 21 of 26 LAU

# Inverted Slider-Crank

Using half angles:

$$P \sin \theta_4 + Q \cos \theta_4 + R = 0$$

$$(R - Q) \tan^2(\theta_4/2) + 2P \tan(\theta_4/2) + (Q + R) = 0$$

Let  $S = R - Q$

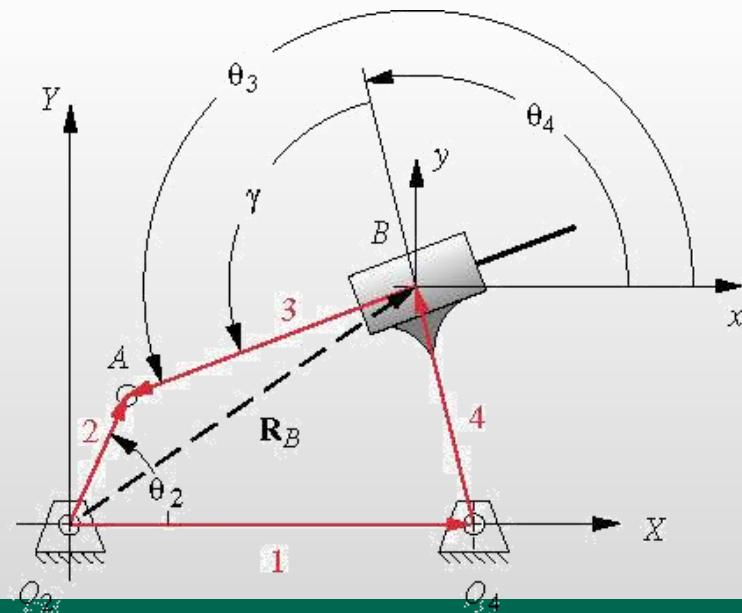
$$T = 2P$$

$$U = Q + R$$

**Solution**

$$\theta_4 = 2\arctan \left( \frac{(-T \pm \sqrt{T^2 - 4SU})}{2S} \right)$$

This has both **open & crossed** solutions

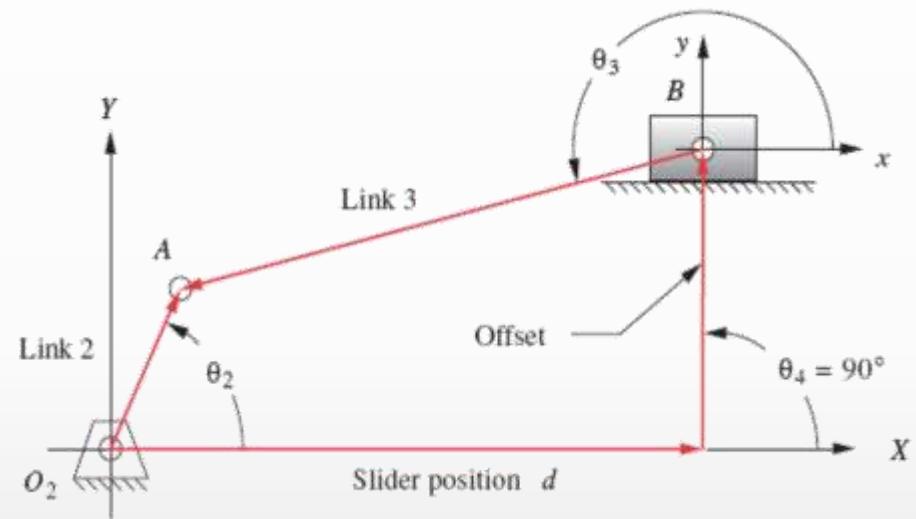


# Inverted Slider-Crank Having $\theta_4$ solve for $\theta_3$ and b **Open configuration** $$\theta_3 = \theta_4 + \gamma$$ $$b = (a \sin \theta_2 - c \sin \theta_4) / \sin \theta_3$$ **Crossed configuration** $$\theta_3 = \theta_4 - \gamma$$ $$b = (a \sin \theta_2 - c \sin \theta_4) / \sin \theta_3$$ The diagram illustrates the geometry of an inverted slider-crank mechanism. It features a fixed pivot point at the origin (0,0) where link 1 is attached. Link 1 is connected to a slider block A, which moves linearly along the horizontal X-axis. Link 2 is connected to slider A and rotates about it. Link 3 is a horizontal connecting rod connecting the end of link 2 to the midpoint of link 4. Link 4 is a crank that rotates about a fixed pivot point O<sub>4</sub> located below the X-axis. The angle between link 2 and link 3 is θ<sub>2</sub>. The angle between link 3 and link 4 is θ<sub>3</sub>. The angle between link 4 and the horizontal is θ<sub>4</sub>. The distance from the center of O<sub>4</sub> to the midpoint of link 4 is labeled R<sub>B</sub>. The vertical distance from the X-axis to the center of O<sub>4</sub> is labeled b. The diagram shows two configurations: 'Open' (top) where link 4 is positioned such that its extension does not cross link 3, and 'Crossed' (bottom) where link 4 is positioned such that its extension crosses link 3. In both configurations, the angle γ is shown as the angle between the extension of link 3 and link 4. MEE341 – Lecture 11: Analytical Linkage Synthesis Slide 23 of 26 LAU

# Example - Slider Crank

**Given:**

- Link 2 = 1.4"
- Link 3 = 4"
- Offset = 1"
- $\theta_2 = 45^\circ$



**Find:**  $\theta_3$  &  $d$

# Example - Slider Crank Solution

Use formulas derived before

**1st configuration (crossed)**

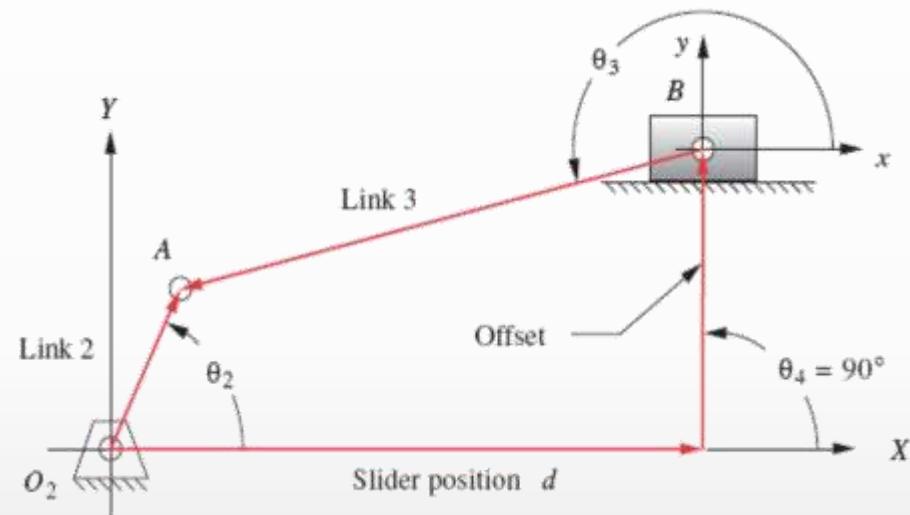
$$\theta_{3_1} = \sin^{-1}\left(\frac{a \sin \theta_2 - c}{b}\right) = -0.14^0$$

$$d = a \cos \theta_2 - b \cos \theta_3 = -3.01 \text{ in}$$

**2nd configuration (open)**

$$\theta_{3_2} = \sin^{-1}\left(-\frac{a \sin \theta_2 - c}{b}\right) + \pi = 180.14^0$$

$$d = a \cos \theta_2 - b \cos \theta_3 = 4.99 \text{ in}$$



*a = length of link 2*

*b = length of link 3*

*c = length of link 4*

*d = length of link 1*

# Solution

