Kinematics & Dynamics of Linkages

Lecture 11 – Analytical Linkage Synthesis

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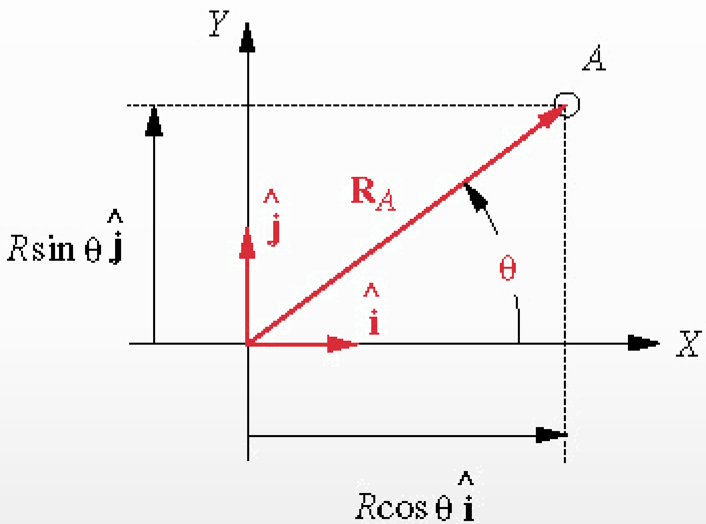
LAU

Algebraic analysis

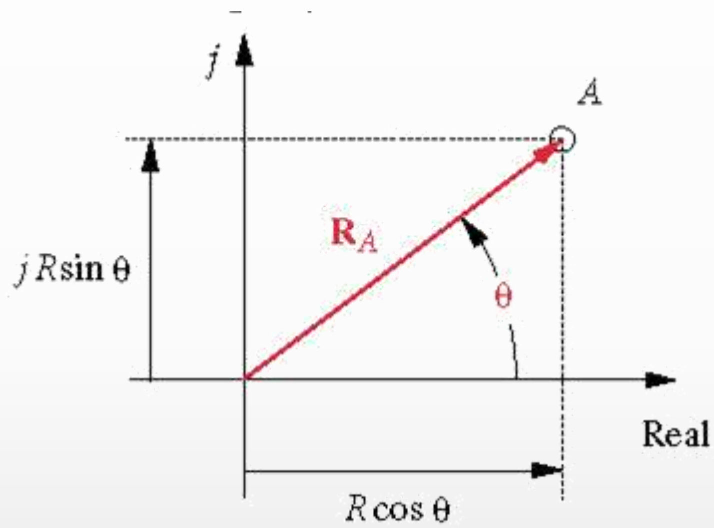


Polar form:

$$|\mathbf{R}_A| \; @ \angle \theta$$

Cartesian form:

$$R\cos\theta\,\hat{\mathbf{i}}\,,\; R\sin\theta\,\hat{\mathbf{j}}$$



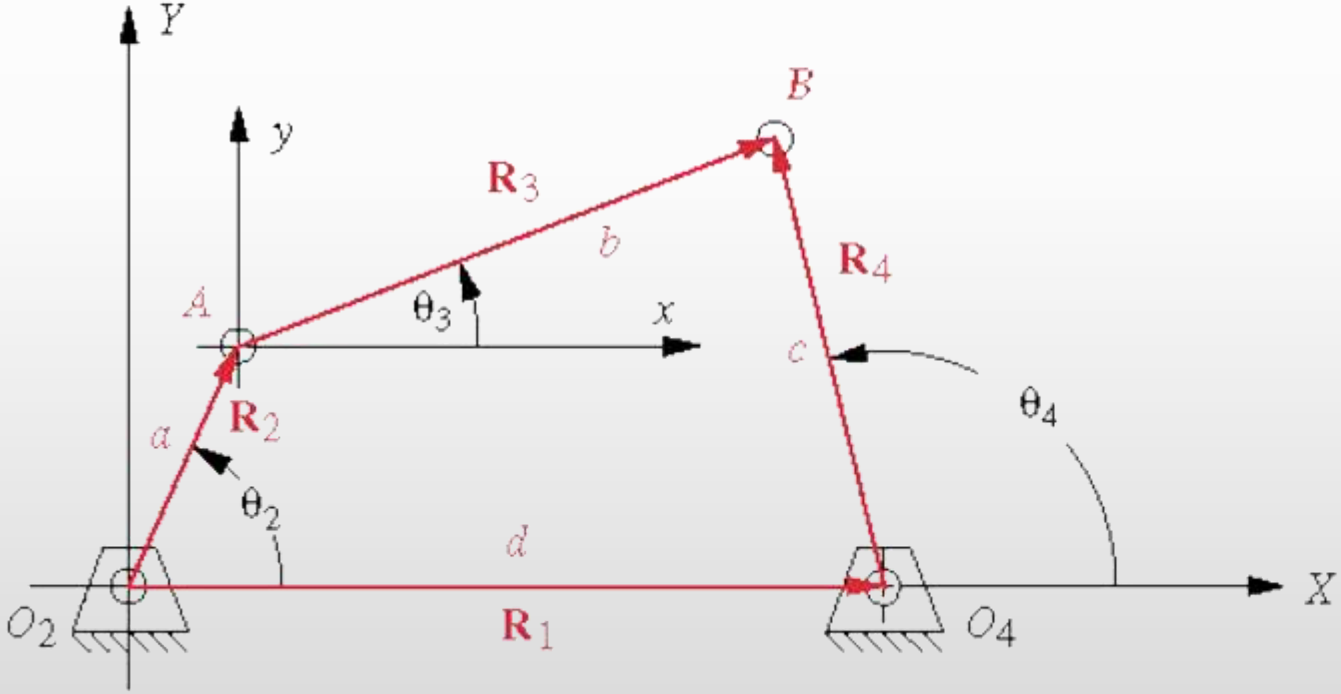
Polar form: $R\,e^{j\theta}$

Cartesian form: $R\cos\theta + jR\sin\theta$

$$R = |\mathbf{R}_A|$$

Vector Loop Approach

$$R_2 + R_3 - R_4 - R_1 = 0$$



Equations

$\mathbf{R_2 + R_3 - R_4 - R_1 = 0}$

Real part

$a \cos\theta_2 + b \cos\theta_3 - c \cos\theta_4 - d = 0$

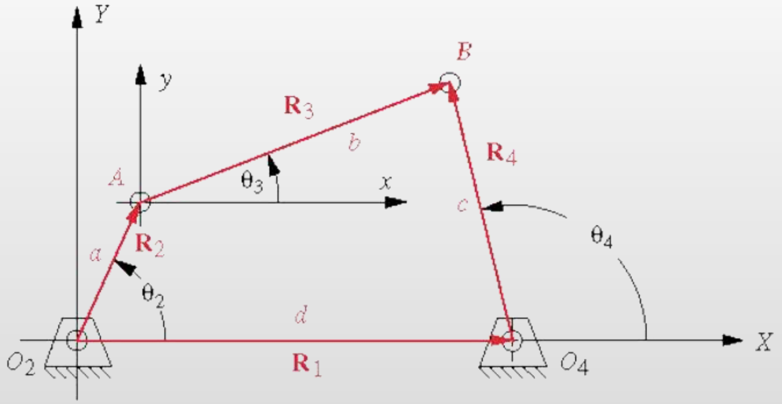
Imaginary part

$a \sin\theta_2 + b \sin\theta_3 - c \sin\theta_4 = 0$

Goal is to find

$\theta_3 = f(a,b,c,d,\theta_2)$

$\theta_4 = g(a,b,c,d,\theta_2)$



4Bar Solution Derivation

Isolate θ_3

$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d$ $\quad$ $b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4$

Square both sides

$(b\cos\theta_3)^2 = (-a\cos\theta_2 + c\cos\theta_4 + d)^2$ $\quad$ $(b\sin\theta_3)^2 = (-a\sin\theta_2 + c\sin\theta_4)^2$

Add the 2 expressions

$$b^2(\cos^2\theta_3 + \sin^2\theta_3) = (-a\cos\theta_2 + c\cos\theta_4 + d)^2 + (-a\sin\theta_2 + c\sin\theta_4)^2$$

So this yields (using $\cos^2\theta_3 + \sin^2\theta_3 = 1$)

$$b^2 = (-a\cos\theta_2 + c\cos\theta_4 + d)^2 + (-a\sin\theta_2 + c\sin\theta_4)^2$$

4Bar Solution Derivation

Multiply & combine like terms

$$b^2 = a^2 + c^2 + d^2 - 2ad\cos\theta_2 + cd\cos\theta_4 - 2ac\cos(\theta_2 - \theta_4)$$

Divide both sides by 2ac

$$-(-b^2+a^2+c^2+d^2)/2ac + (d/c)\cos\theta_2 - (d/a)\cos\theta_4 = -\cos(\theta_2 - \theta_4)$$

To simplify, define

$$k_1 = (d/a)$$
$$k_2 = (d/c)$$
$$k_3 = (a^2 - b^2 + c^2 + d^2)/2ac$$

Yields Freudenstein's equation

$$k_1\cos\theta_4 - k_2\cos\theta_2 + k_3 = \cos(\theta_2 - \theta_4) = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4$$

4Bar Solution Derivation

Yields Freudenstein's equation

$k_1\cos\theta_4 - k_2\cos\theta_2 + k_3 = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4$

From Trigonometric relations:

$\sin\theta_4 = (2\tan(\theta_4/2))/(1 + \tan^2(\theta_4/2))$

$\cos\theta_4 = (1 - \tan^2(\theta_4/2))/(1 + \tan^2(\theta_4/2))$

Substituting $A\tan^2(\theta_4/2) + B\tan(\theta_4/2) + C = 0$

where

$A = \cos\theta_2 - k_1 - k_2\cos\theta_2 + k_3$

$B = -2\sin\theta_2$

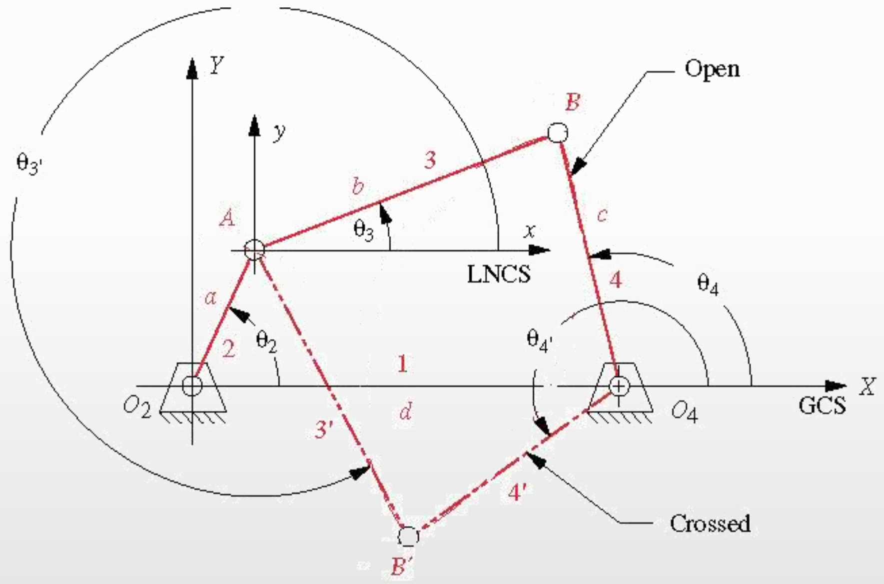
$C = k_1 - (k_2 + 1)\cos\theta_2 + k_3$

4Bar Solution Derivation

Solution of $A\tan^2(\theta_4/2) + B\tan(\theta_4/2) + C = 0$

$$\tan\left(\frac{\theta_4}{2}\right) = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}$$

$$\theta_4 = 2\tan^{-1}\left(\frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}\right)$$



We have **2 solutions** because we have 2 possible linkage configurations (**Open** and **Crossed**)

4Bar Solution Derivation

Solving for θ_3:

Original loop equations

$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0 \qquad a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 = 0$

Isolate θ_4

$c\cos\theta_4 = a\cos\theta_2 + b\cos\theta_3 - d \qquad c\sin\theta_4 = a\sin\theta_2 + b\sin\theta_3$

Square both sides and add the equations to eliminate θ_4

$k_1\cos\theta_3 - k_4\cos\theta_2 + k_5 = \cos(\theta_2 - \theta_3) = \cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3$

Where

$k_1 = (d/a)$
$k_4 = (d/b)$
$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab$

4Bar Solution Derivation

Similarly, we may derive a quadratic equation for tan ($\theta_3/2$)

$$D \tan^2(\theta_3/2) + E \tan(\theta_3/2) + F = 0$$

Where

$$D = \cos\theta_2 - k_1 + k_4\cos\theta_2 + k_5$$

$$E = -2\sin\theta_2$$

$$F = k_1 + (k_4 - 1)\cos\theta_2 + k_5$$

$$k_1 = (d/a)$$

$$k_4 = (d/b)$$

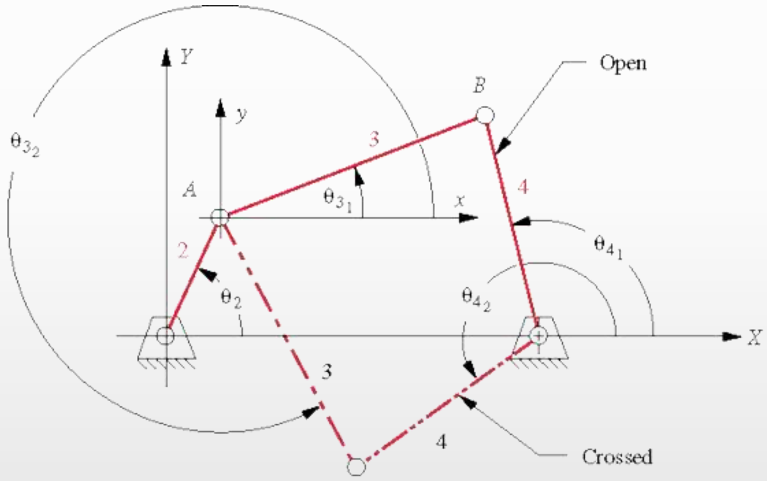
$$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab$$

4Bar Solution Derivation

The solution to the quadratic is

$$\tan\left(\frac{\theta_3}{2}\right) = \frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D}$$

$$\theta_3 = 2\tan^{-1}\left(\frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D}\right)$$



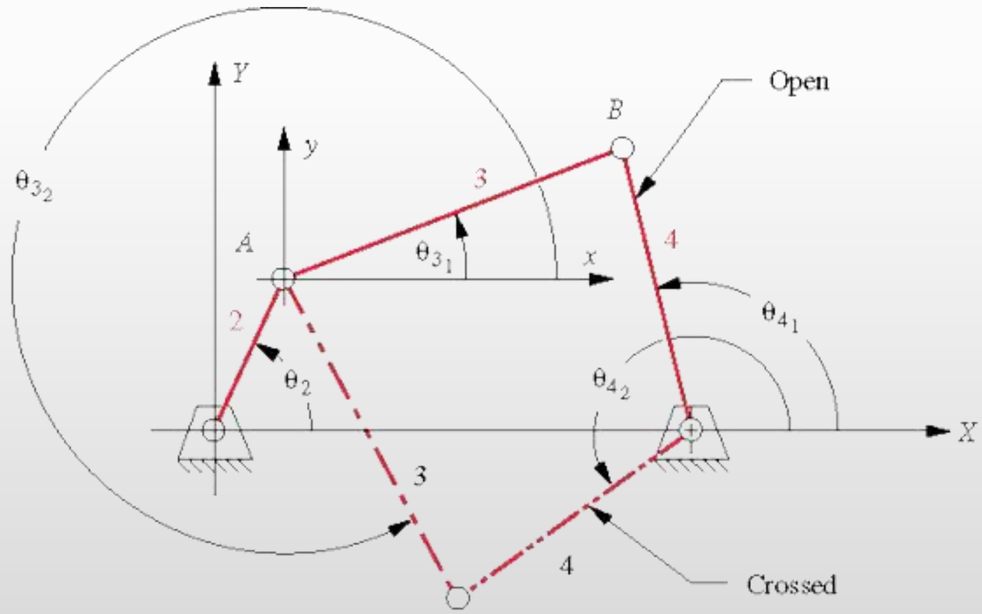
Again, **2 solutions** for **open** & **crossed linkages**

Example

- Link 1 = 8″ Link 3 = 8″
- Link 2 = 5″ Link 4 = 6″

- θ2 = 75º

- Calculate θ3 & θ4 ??



Example - Solution

To determine θ_3 & θ_4 first calculate the constants

$k_1 = (d/a) = 1.6$
$k_2 = (d/c) = 1.333$
$k_3 = (a^2 - b^2 + c^2 + d^2)/2ac = 1.017$
$k_4 = (d/b) = 1.0$
$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab = -1.463$

a = length of link 2
b = length of link 3
c = length of link 4
d = length of link 1

$A = \cos\theta_2 - k_1 - k_2\cos\theta_2 + k_3 = -0.669$
$B = -2\sin\theta_2 = -1.932$
$C = k_1 - (k_2 + 1)\cos\theta_2 + k_3 = 2.013$
$D = \cos\theta_2 - k_1 + k_4\cos\theta_2 + k_5 = -2.545$
$E = -2\sin\theta_2 = -1.932$
$F = k_1 + (k_4 - 1)\cos\theta_2 + k_5 = 0.137$

Example - Solution

The solution of $\boldsymbol{\theta_4}$:

$$\theta_4 = 2\tan^{-1}\left(\frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}\right)$$ **= 78.2° or –149.7°**

The solution of $\boldsymbol{\theta_4}$:

$$\theta_3 = 2\tan^{-1}\left(\frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D}\right)$$ **= 7.5° or –79.0°**

Slider-Cranks

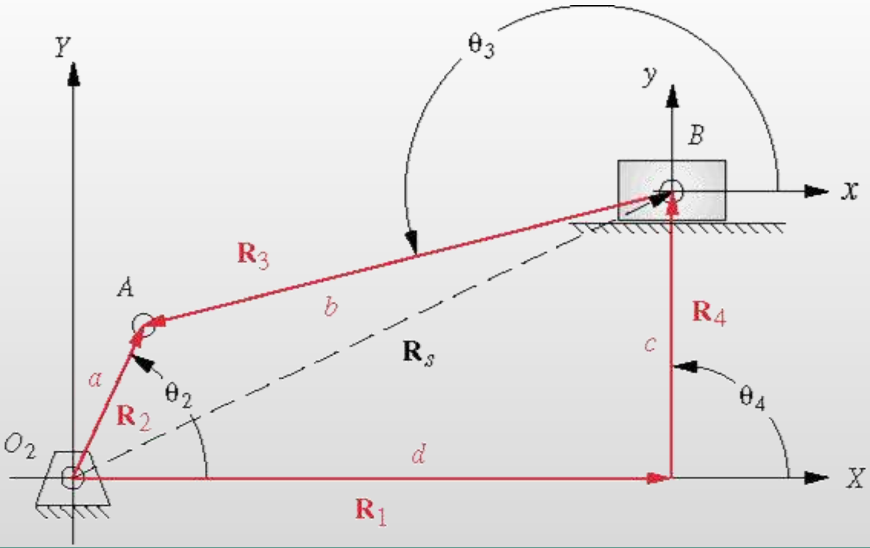
Vector Loop Approach

Given:

- Lengths of links 2 & 3
- Offset height
- Input angle θ_2

Find:

- Length of link 1
- Angle θ_3



Slider–Cranks

Vector loop equation $R_2 - R_3 - R_4 - R_1 = 0$

Projections: (x and y axis)

$a \cos\theta_2 - b \cos\theta_3 - c \cos\theta_4 - d = 0$

$a \sin\theta_2 - b \sin\theta_3 - c \sin\theta_4 = 0$

Where

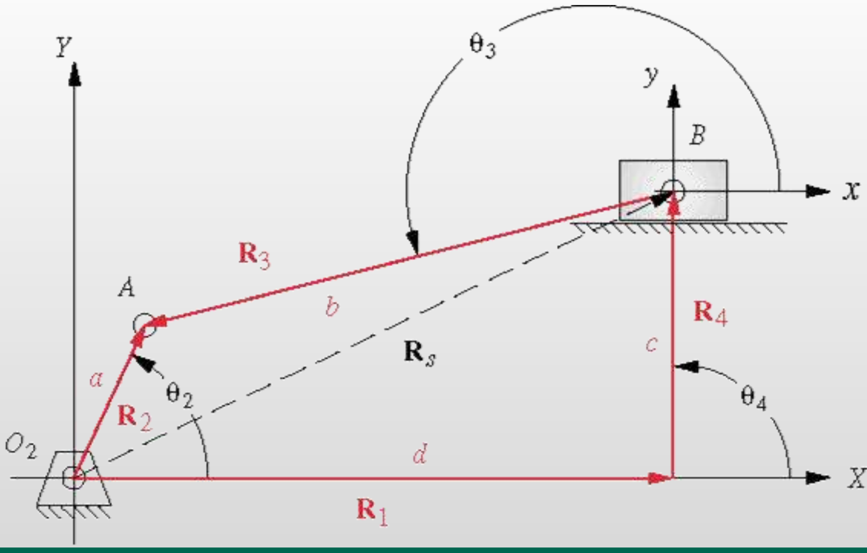
a, b and c are known

θ_2 is given

$\theta_4 = 90°$

Solution

$\mathbf{d} = a \cos\theta_2 - b \cos\theta_3$



Slider–Cranks

Solve for θ_3 using the equation (projection on y axis)

$a \sin\theta_2 - b \sin\theta_3 - c \sin\theta_4 = 0$

<u>Solution</u> (± 90^0 of sin^{-1})

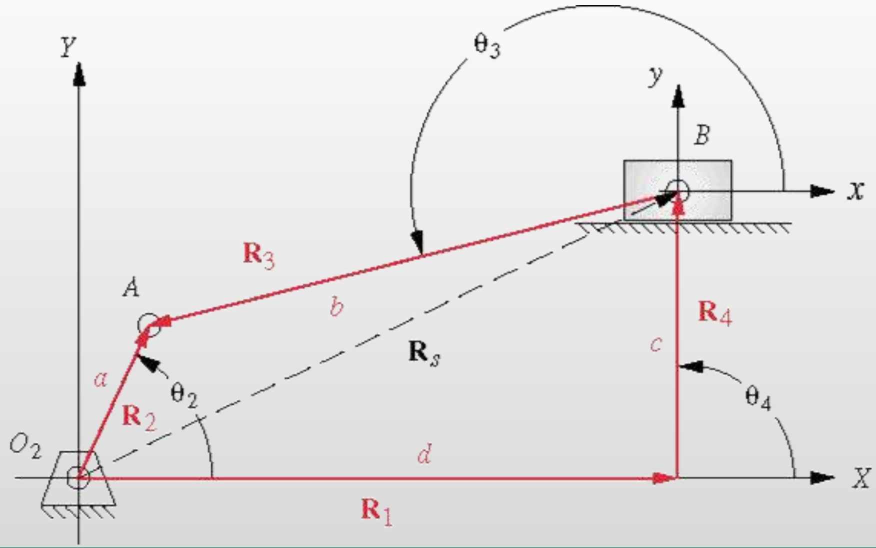
1st configuration

$$\theta_{3_1} = \sin^{-1}\left(\frac{a \sin\theta_2 - c}{b}\right)$$

2nd configuration

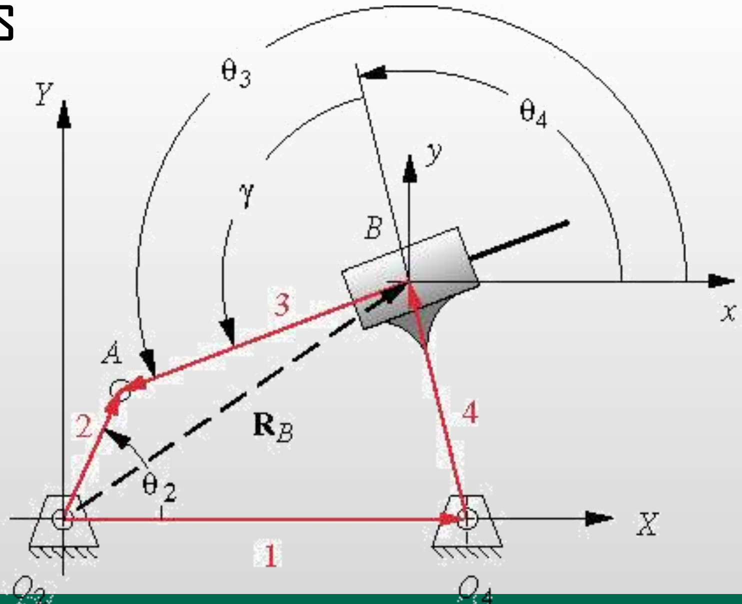
$$\theta_{3_2} = \sin^{-1}\left(-\frac{a \sin\theta_2 - c}{b}\right) + \pi$$

d = $a \cos\theta_2 - b \cos\theta_3$



Inverted Slider–Cranks

- Sliding joint between links 3 & 4 at point B
- γ does not have to be 90°
- All slider–cranks will have at least one link whose effective length between joints will vary as the linkage moves



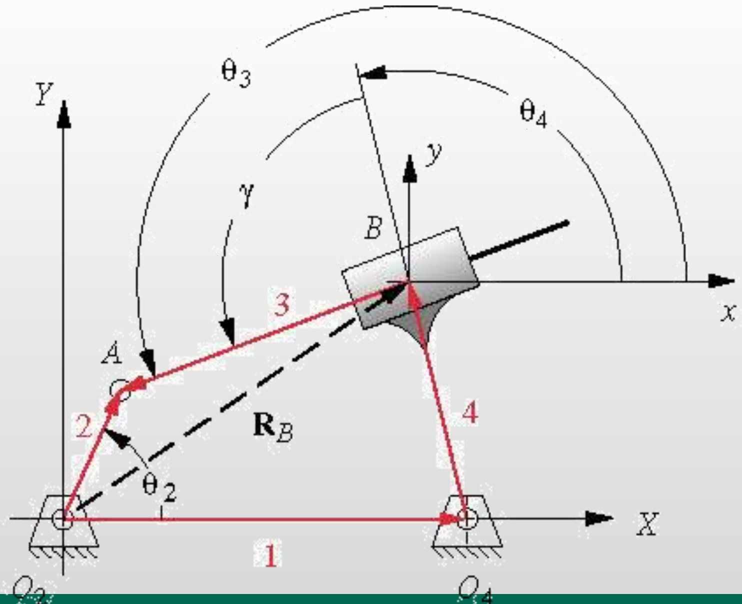
Inverted Slider–Cranks

- **Vector Loop Approach**
- Length of link 3 (b) changes with time
    - b is unknown
    - θ_4 is unknown
    - θ_3 is unknown

- Relationship between θ_3 and θ_4

$$\theta_3 = \theta_4 \pm \gamma$$

+ for an open linkage
– for a crossed linkage



Inverted Slider-Cranks

Vector loop equation: $R_2 - R_3 - R_4 - R_1 = 0$

Original loop equations

$a \cos\theta_2 - b \cos\theta_3 - c \cos\theta_4 - d = 0$

$a \sin\theta_2 - b \sin\theta_3 - c \sin\theta_4 = 0$

Where a, d and c are known, θ_2 is given, θ_4, θ_3 and b to be calculated

Use the second equation to solve for b

$b = (a \sin\theta_2 - c \sin\theta_4) / \sin\theta_3$

Substituting into the first equation

$a \cos\theta_2 - ((a \sin\theta_2 - c \sin\theta_4)/ \sin\theta_3) \cos\theta_3 - c \cos\theta_4 - d = 0$

Inverted Slider–Cranks

Rearrange equation

$a \cos\theta_2 - ((a \sin\theta_2 - c \sin\theta_4)/ \sin\theta_3) \cos\theta_3 - c \cos\theta_4 - d = 0$

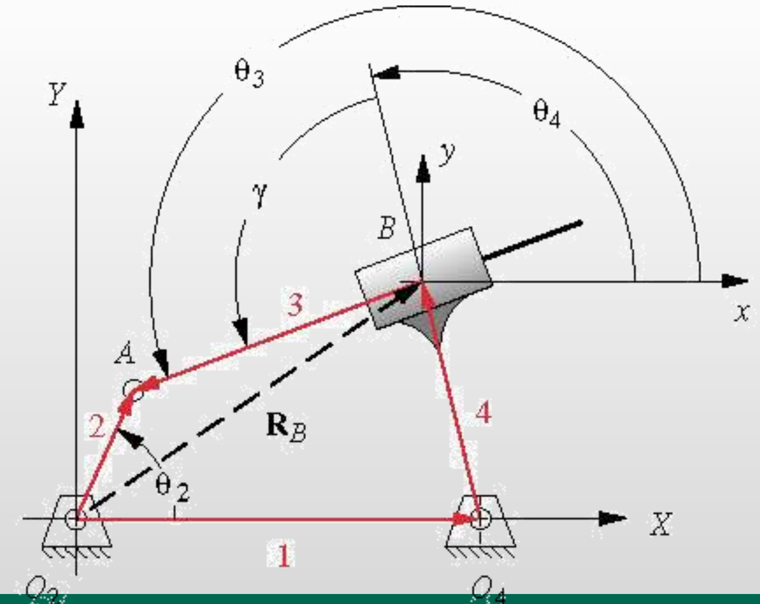
Using: $\theta_3 = \theta_4 \pm \gamma$

Yields: $P \sin\theta_4 + Q \cos\theta_4 + R = 0$

Where $P = a \sin\theta_2 \sin\boldsymbol{\gamma} + (a \cos\theta_2 - d) \cos\boldsymbol{\gamma}$

$Q = - a \sin\theta_2 \cos\boldsymbol{\gamma} + (a \cos\theta_2 - d) \sin\boldsymbol{\gamma}$

$R = - c \sin\boldsymbol{\gamma}$



Inverted Slider–Cranks

Using half angles:

$$P \sin\theta_4 + Q \cos\theta_4 + R = 0$$

$$(R - Q) \tan^2 (\theta_4/2) + 2P \tan(\theta_4/2) + (Q + R) = 0$$

Let

$S = R - Q$

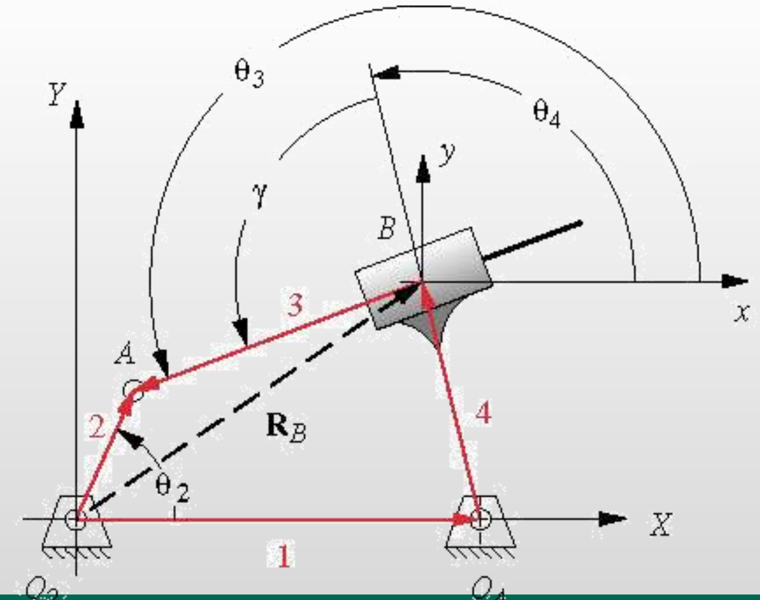
$T = 2P$

$U = Q + R$

Solution

$\theta_4 = 2\arctan\left(\left(- T \pm (T^2 - 4SU)^{.5}\right)/2S\right)$

This has both **open & crossed** solutions



Inverted Slider–Cranks

Having θ_4 solve for θ_3 and b

Open configuration

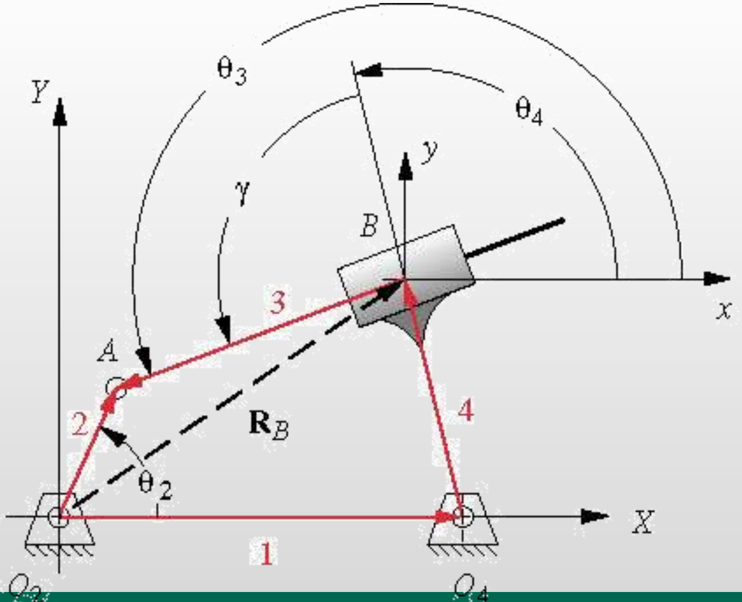
$\theta_3 = \theta_4 + \gamma$

$b = (a \sin\theta_2 - c \sin\theta_4) / \sin\theta_3$

Crossed configuration

$\theta_3 = \theta_4 - \gamma$

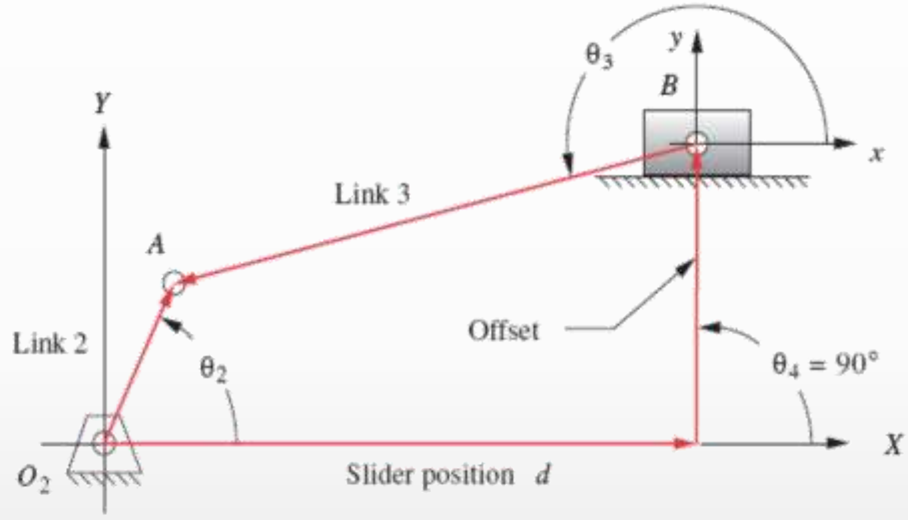
$b = (a \sin\theta_2 - c \sin\theta_4) / \sin\theta_3$



Example – Slider Crank

Given:

- Link 2 = 1.4"
- Link 3 = 4"
- Offset = 1"
- $\theta_2 = 45^o$



Find: θ_3 & d

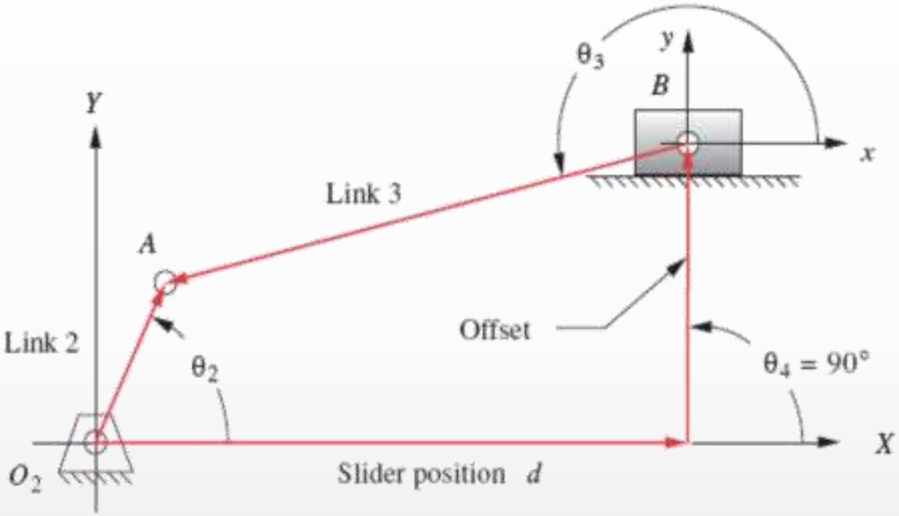
Example – Slider Crank Solution

Use formulas derived before

1st configuration (crossed)

$$\theta_{3_1} = \sin^{-1}\left(\frac{a\sin\theta_2 - c}{b}\right) = -0.14^0$$

$$d = a\cos\theta_2 - b\cos\theta_3 = -3.01\,in$$



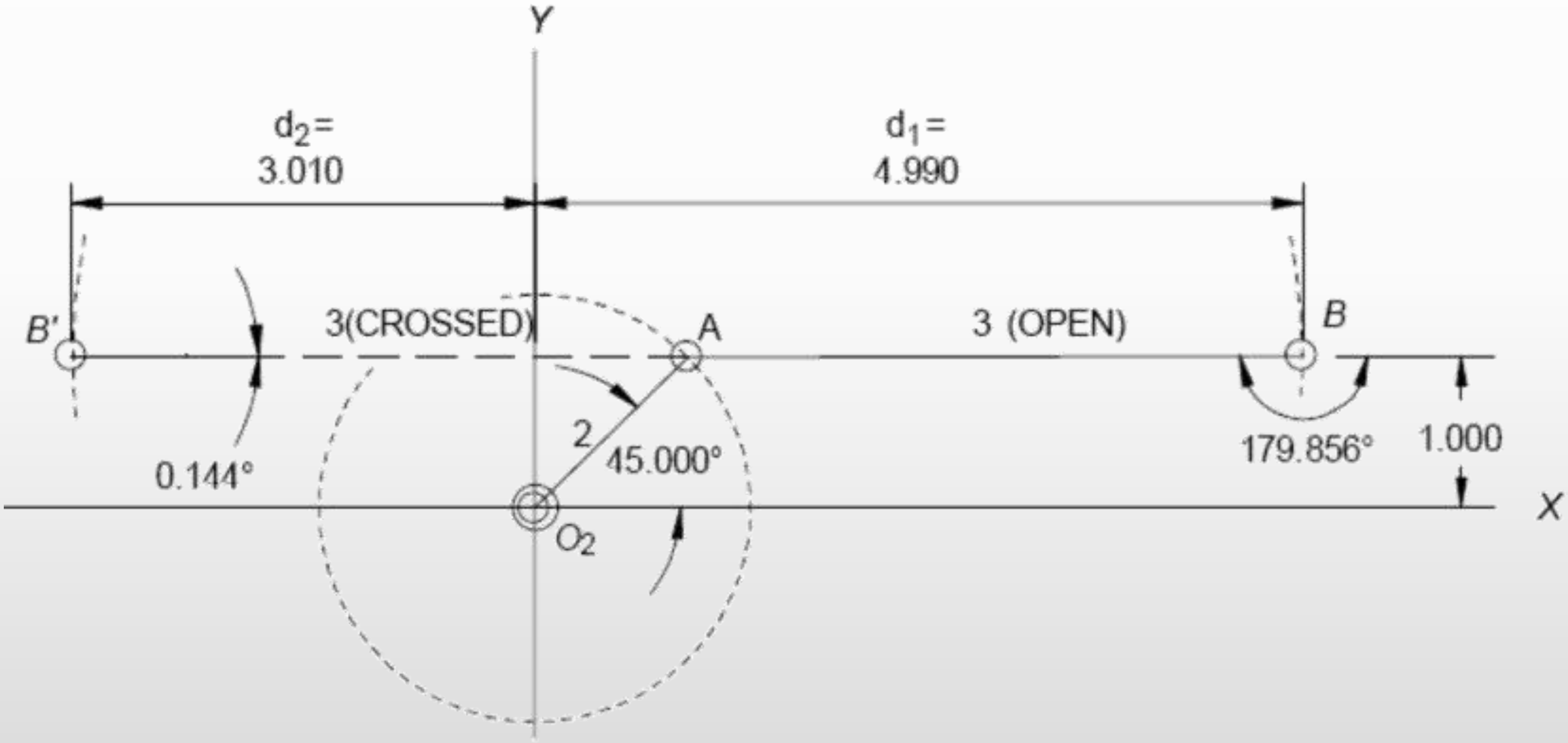
2nd configuration (open)

$$\theta_{3_2} = \sin^{-1}\left(-\frac{a\sin\theta_2 - c}{b}\right) + \pi = 180.14^0$$

$$d = a\cos\theta_2 - b\cos\theta_3 = 4.99\,in$$

a *= length of link 2*
b *= length of link 3*
c *= length of link 4*
d *= length of link 1*

Solution



Y
d2=
3.010
d1=
4.990
B'
3(CROSSED)
A
3 (OPEN)
B
2
45.000°
0.144°
179.856°
1.000
X
O2